

THE CONSTRAINED KRASNOSELS'KII FORMULA FOR PARABOLIC DIFFERENTIAL INCLUSIONS

WOJCIECH KRYSZEWSKI AND JAKUB SIEMIANOWSKI

ABSTRACT. We consider a constrained evolution inclusions of parabolic type (1) involving an m -dissipative linear operator and the source term of multivalued type in a Banach space and topological properties of the solution map. We show a relation between the constrained fixed point index of the Krasnosel'skii–Poincaré operator of translation along trajectories associated with (1) and the appropriately defined constrained degree of $A + F(0, \cdot)$ of the right-hand side in (1). Our results extend those of [11] and [14].

1. INTRODUCTION

We study the initial value problem for a semilinear differential inclusion

$$(1) \quad \begin{cases} \dot{u}(t) \in Au(t) + F(t, u(t)), & t \in [0, 1], \ u \in \mathcal{K}, \\ u(0) = x \in \mathcal{K}, \end{cases}$$

where \mathbb{E} is a Banach space, $\mathcal{K} \subset \mathbb{E}$ is a closed convex set of state constraints, $A : D(A) \subset \mathbb{E} \rightarrow \mathbb{E}$ generates a compact strongly continuous linear semigroup $\{S(t)\}_{t \geq 0}$ on \mathbb{E} and $F : [0, 1] \times \mathcal{K} \rightrightarrows \mathbb{E}$ is a set-valued map. A continuous $u : [0, 1] \rightarrow \mathbb{E}$ is a (mild) *solution* to (1) if it stays in \mathcal{K} , i.e., $u(t) \in \mathcal{K}$ and

$$u(t) = S(t)x + \int_0^t S(t-s)w(s)ds$$

for all $t \in [0, 1]$, where $w \in L^1([0, 1], \mathbb{E})$ and $w(s) \in F(s, u(s))$ a.e. on $[0, 1]$.

The study of (1) is justified and motivated by a partial differential inclusion of parabolic type

$$(*) \quad \begin{cases} u_t - \Delta u \in \varphi(t, x, u), & t \in [0, 1], \ x \in \Omega, \ u \in K \\ u(0, \cdot) = g = (g_1, \dots, g_N) \in L^2(\Omega, \mathbb{R}^N), \ g(x) \in K \text{ for a.e. } x \in \Omega, \\ u|_{[0, 1] \times \partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^M$ is a bounded domain with smooth boundary, $K \subset \mathbb{R}^N$ is convex closed and $\varphi : [0, 1] \times \Omega \times K \rightrightarrows \mathbb{R}^N$ is upper semicontinuous with convex compact values. Generalized systems of the form (*) model reaction-diffusion processes with uncertain reaction term or (via set-valued regularization) those with discontinuous reaction term. We are looking for (strong) solutions with values in K , i.e. $u = (u_1, \dots, u_N) : [0, 1] \times \Omega \rightarrow \mathbb{R}^N$ such that $u(t, x) \in K$ for all $t \in [0, 1]$, $x \in \Omega$, $u_i(t, \cdot) \in H^1 \cap H^2(\Omega)$ for a.a. $t \in [0, 1]$, the function $t \mapsto h_i(t) := \Delta u_i(t, \cdot)$ belongs to $L^1([0, 1], L^2(\Omega))$ and $u_i(t, \cdot) = g + \int_0^t (h(s) + w_i(s))ds$ for all $i = 1, \dots, N$, where $w = (w_1, \dots, w_N) : [0, 1] \rightarrow L^2(\Omega, \mathbb{R}^N)$ is integrable and $w(s)(x) \in \varphi(s, x, u(s, x))$ a.e. for $s \in [0, 1]$ and $x \in \Omega$. The role of the constraining set K may be explained as follows: treating u_i as the concentration of the i -th among N components under diffusion, one has $u_i \geq 0$ since concentration cannot be negative. On the other hand, there is an upper bound, say $u_i(t, x) \leq R_i$ on $[0, 1] \times \Omega$, beyond which the i -th component is saturated. Thus, the natural question is to study the existence and behavior of solutions $u = (u_1, \dots, u_N)$ in the cube $[0, R_1] \times \dots \times [0, R_N]$. This is just a heuristic simplification, and so, instead of the cube, we consider an arbitrary closed convex set K .

2010 *Mathematics Subject Classification.* 34A60, 34K09, 35K58, 47H04, 47H11, 47J35, 54C60, 55M20.

Key words and phrases. Krasnosel'skii–Poincaré operator, evolution differential inclusion, topological degree, set-valued map.

In order to get solutions to (1) we will rely on the semigroup invariance of \mathcal{K} and the *weak tangency* condition:

$$(2) \quad F(t, x) \cap T_{\mathcal{K}}(x) \neq \emptyset \text{ for all } t \in [0, 1], x \in \mathcal{K},$$

where

$$(3) \quad T_{\mathcal{K}}(y) := \text{cl} \bigcup_{h>0} h^{-1}(\mathcal{K} - x) = \left\{ v \in \mathbb{E} \mid \lim_{h \rightarrow 0^+} \frac{1}{h} d(y + hv, \mathcal{K}) = 0 \right\}$$

stands for the *tangent cone* to \mathcal{K} at $y \in \mathcal{K}$ (cl stands for the closure and $d(z, \mathcal{K})$ is the distance of $z \in \mathbb{E}$ to \mathcal{K}). These conditions, being in fact too strong for the existence only, are very well-justified and, moreover, imply the R_δ -structure of the set of all solutions to (1) and allow to compare the fixed point index of the Poincaré t -operator $\Sigma_t : \mathcal{K} \rightarrow \mathcal{K}$, $t > 0$, associated with (1) given by

$$\Sigma_t(x) := \{u(t) \in \mathcal{K} \mid u \text{ is a solution of } (*), u(0) = x\}, x \in \mathcal{K},$$

with the below introduced constrained topological degree of the right-hand side $A + F(0, \cdot)$. In this way we obtain a generalization of the celebrated Krasnosel'skii formula.

Recall that the classical Krasnosel'skii formula concerns an ODE $\dot{x} = f(t, x)$, $x \in \mathbb{R}^N$, $t \in [0, 1]$, with locally Lipschitz $f : [0, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, admitting global solutions. If $U \subset \mathbb{R}^N$ is open bounded and $f(x, 0) \neq 0$ for x in the boundary ∂U of U , then the Brouwer degrees $\deg_B(-f(0, \cdot), U) = \deg_B(I - P_t, U)$, where P_t is the associated Poincaré operator (cf. [22, Lem. 13.1., 13.2.]). An infinite dimensional variant of the Krasnosel'skii formula was obtained in [11] in the case of (1) with single-valued, time-independent and locally Lipschitz nonlinearity F and in the context of bifurcation results in [14], where the unconstrained situation was considered.

After this introduction the paper is organized as follows: in the second section we introduce the notation along with some auxiliary lemmata; in the third one we discuss in detail assumptions on A , \mathcal{K} and F in (1) and show that they are motivated and follow directly from the natural and mild hypotheses concerning (*). In the fourth section we establish the R_δ -structure of solutions to (1) and, in the fifth one the appropriate degree of the right-hand side in (1) is defined. In the final, sixth section we prove the announced Krasnosel'skii formula.

2. PRELIMINARIES

In what follows $(\mathbb{E}, \|\cdot\|)$ denotes a real Banach space, while \mathbb{E}^* is the normed topological dual of \mathbb{E} ; we write $\langle x, p \rangle$ instead of $p(x)$ for $x \in \mathbb{E}, p \in \mathbb{E}^*$; $\mathcal{L}(\mathbb{E})$ denotes the space of bounded linear operators on \mathbb{E} . By $L^1([0, T], \mathbb{E})$ (resp. $\mathcal{C}([0, 1], \mathbb{E})$) we denote the space of Bochner integrable (resp. continuous) functions $u : [0, T] \rightarrow \mathbb{E}$. Recall that $A \subset L^1([0, 1], \mathbb{E})$ is *integrably bounded* if there exists $\lambda \in L^1([0, 1], \mathbb{R})$ such that $\|\alpha(t)\| \leq \lambda(t)$ a.e. for every $\alpha \in A$. If X is a metric space, $\varepsilon > 0$ then $B_X(A, \varepsilon) := \{x \in X \mid d(x; A) := \inf_{a \in A} d(x, a) < \varepsilon\}$. If $X \subset \mathbb{E}$, Y is a topological space, then a continuous $f : X \rightarrow Y$ is *compact* or *completely continuous* if $f(B)$ is relatively compact for each bounded $B \subset X$.

A *set-valued* map $\varphi : X \rightarrow Y$ assigns to each $x \in X$ a nonempty subset $\varphi(x) \subset Y$. If X, Y are topological spaces, then φ is *upper semicontinuous* or usc (resp. *lower semicontinuous* or lsc) if $\varphi^{-1}(A) := \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}$ is closed (resp. open) for every closed $A \subset Y$. If $X \subset \mathbb{E}$, then $\varphi : X \rightarrow Y$ is *compact* if it is usc and $\varphi(B) := \bigcup_{x \in B} \varphi(x)$ is relatively compact for any bounded $B \subset X$. If X, Y are metric spaces, then $\varphi : X \rightarrow Y$ is *H-usc* (resp. *H-lsc*) if for any $x_0 \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\varphi(x) \subset B_Y(\varphi(x_0), \varepsilon)$ (resp. $\varphi(x_0) \subset B_Y(\varphi(x), \varepsilon)$) for $x \in B_X(x_0, \delta)$ (see [15], [16] for details and examples concerning set-valued maps).

We present two results that will be frequently used in a form adopted for our needs. The first one is a simple modification of [5, Lem. 17.].

Lemma 2.1. *Let $\mathcal{K} \subset \mathbb{E}$ be closed convex, $F : [0, 1] \times \mathcal{K} \rightharpoonup \mathbb{E}$ be tangent to \mathcal{K} (see (2)) and H -usc with convex values. For any continuous $\alpha : [0, 1] \times \mathcal{K} \rightarrow (0, \infty)$ there is a locally Lipschitz $f : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ such that $f(t, x) \in T_{\mathcal{K}}(x)$ and*

$$f(t, x) \in F(B_{[0,1]}(t, \alpha(t, x)) \times B_{\mathcal{K}}(x, \alpha(t, x))) + B_{\mathbb{E}}(0, \alpha(t, x)) \quad \text{for } t \in [0, 1], t \in \mathcal{K}. \quad \square$$

If (S, \mathcal{F}) is a measure space, X is a Polish space and Y is a topological space, then $\varphi : S \times X \rightharpoonup Y$ is said to be *product measurable* if, for every open $U \subset Y$, $\varphi^{-1}(U)$ belongs the product σ -algebra $\mathcal{F} \otimes \mathfrak{B}(X)$, where $\mathfrak{B}(X)$ is the Borel σ -algebra in X .

Theorem 2.2. [19, Th. 3.2] *Let \mathbb{E} be a separable Banach space, $\mathcal{K} \subset \mathbb{E}$ be closed convex and let $F, G : [0, 1] \times \mathcal{K} \rightharpoonup \mathbb{E}$ be product measurable (on $[0, 1]$ the Lebesgue σ -algebra is considered) with closed convex values and such that $F(t, x) \cap G(t, x) \neq \emptyset$ for all (t, x) . If $F(t, \cdot)$ is H -usc and $G(t, \cdot)$ is lsc, for $t \in [0, 1]$, then for every $\varepsilon > 0$ there is a Carathéodory map $f : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ (i.e. $f(t, \cdot)$ is continuous for every $t \in [0, 1]$ and $f(\cdot, x)$ is measurable for every $x \in \mathcal{K}$) such that*

$$f(t, x) \in G(t, x) \quad \text{and} \quad f(t, x) \in F(\{t\} \times B_{\mathcal{K}}(x, \varepsilon)) + B_{\mathbb{E}}(0, \varepsilon)$$

for all $t \in [0, 1]$ and $x \in \mathcal{K}$. \square

3. FROM THE SYSTEM OF PDE'S TO AN ABSTRACT PROBLEM

Let us make the following standing assumptions with respect to (1):

- (A) $A : D(A) \rightarrow \mathbb{E}$ generates a compact C_0 -semigroup $\{S(t)\}_{t \geq 0}$ of linear operators on \mathbb{E} ;
- (K) A closed convex $\mathcal{K} \subset \mathbb{E}$ is *semigroup invariant*, i.e. $S(t)(\mathcal{K}) \subset \mathcal{K}$ for every $t \geq 0$;
- (F₁) $F : [0, 1] \times \mathcal{K} \rightharpoonup \mathbb{E}$ has convex weakly compact values;
- (F₂) F is product measurable and for any $t \in [0, 1]$, the map $\mathcal{K} \ni x \mapsto F(t, x) \subset \mathbb{E}$ is H -usc;
- (F₃) there is $c > 0$ such that $\sup_{y \in F(t, x)} \|y\| \leq c(1 + \|x\|)$ for all $t \in [0, 1]$, $x \in \mathcal{K}$;
- (F₄) F is *tangent* to \mathcal{K} , i.e. $F(t, x) \cap T_{\mathcal{K}}(x) \neq \emptyset$ for all $t \in [0, 1]$ and $x \in \mathcal{K}$ (see (3)).

We shall show that these assumptions are consistent with hypotheses usually made with respect to the system (*). But first let us collect some comments.

Remark 3.1. (a) In view of (A) there are $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq M e^{\omega t}$ for $t \geq 0$. For $h > 0$ and $h\omega < 1$, the *resolvent* $J_h := (I - hA)^{-1} : \mathbb{E} \rightarrow D(A) \subset \mathbb{E}$ is well-defined, belongs to $\mathcal{L}(\mathbb{E})$ and $\|J_h\| \leq M(1 - h\omega)^{-1}$ (cf. [27]). By [27, Th. 2.3.3], $\{S(t)\}_{t \geq 0}$ is *compact* (i.e. for any $t > 0$ an operator $S(t) \in \mathcal{L}(\mathbb{E})$ is compact) if and only if it is *resolvent compact*, i.e. for $h > 0$, $h\omega < 1$, J_h is compact and $(0, +\infty) \ni t \mapsto S(t) \in \mathcal{L}(\mathbb{E})$ is continuous.

(b) Assumption (K) means that if the reaction term F vanishes, then the diffusion process $u(t) = S(t)x$, $x \in \mathcal{K}$, survives in \mathcal{K} . It holds if and only if $J_h(\mathcal{K}) \subset \mathcal{K}$ for $h > 0$ with $h\omega < 1$ (comp. [21, Sec. 3.1.] and cf. [11, Rem. 4.6.]).

(c) Assumptions (F₁)-(F₂) together with [6, Proposition 2.3] imply that for all $t \in [0, 1]$ the set-valued map $F(t, \cdot) : \mathcal{K} \rightharpoonup \mathbb{E}$ is *weakly usc*, i.e. usc with respect to the original topology in \mathcal{K} and the weak topology in \mathbb{E} . In particular, for each $t \in [0, 1]$ the image $F(\{t\} \times D) \subset \mathbb{E}$ of a compact subset $D \subset \mathcal{K}$ is weakly compact. Moreover, since values of F are convex, we gather that the graph of $F(t, \cdot)$ is closed in $\mathcal{K} \times \mathbb{E}$, where the original topology in \mathcal{K} and the weak topology in \mathbb{E} are considered, i.e., if $x_n \rightarrow x$ in \mathcal{K} , $y_n \in F(t, x_n)$ and $y_n \rightharpoonup y$ (weakly), then $y \in F(t, x)$. Condition (F₃) implies the global (unconstrained) existence of solutions.

(d) It is easy to see that (K) implies that for all $x \in \mathcal{K}$

$$T_{\mathcal{K}}(x) \subset T_{\mathcal{K}}^A(x) := \left\{ v \in \mathbb{E} \left| \liminf_{t \rightarrow 0^+} \frac{1}{t} d(S(t)x + tv, \mathcal{K}) = 0 \right. \right\}.$$

Hence (K) together with (F_4) imply that

$$(4) \quad F(t, x) \cap T_{\mathcal{K}}^A(x) \neq \emptyset, \quad x \in \mathcal{K}, \quad t \in [0, 1].$$

The sets $T_{\mathcal{K}}^A(x)$, $x \in \mathcal{K}$, have been introduced by Pavel [25] and condition (4) shown to be necessary and sufficient for the existence of (mild) solutions surviving in \mathcal{K} of (1), when F single-valued continuous. This condition is also sufficient for the existence in case of a H-usc set-valued perturbation F (see [7, §4.5] and [26]); see also [9, Chap. 9] for a detailed discussion of different tangency issues. Our study of (K) along with (F_4) is motivated by Proposition 3.3, the second part of Proposition 3.4 and Remark 3.2.

Let us now return to $(*)$ and make the following assumptions:

(φ_1) $\varphi : [0, 1] \times \Omega \times K \multimap \mathbb{R}^N$ is usc with convex compact values;

(φ_2) $\sup_{v \in \varphi(t, x, y)} |v| \leq \alpha(x) + c|y|$, for some $\alpha \in L^2(\Omega)$, $c > 0$ and for all $t \in [0, 1]$, $x \in \Omega$ and $y \in K$.

(φ_3) φ is *tangent* to K , i.e. $\varphi(t, x, y) \cap T_K(y) \neq \emptyset$ for $t \in [0, 1]$, $x \in \Omega$ and $y \in K$ ⁽¹⁾

Remark 3.2. In order to understand the physical meaning of (φ_3) consider an important special case when $K = \mathbb{R}_+^N$, i.e., $u = (u_1, \dots, u_N) \in K$ if and only if $u_i \geq 0$ for $i = 1, \dots, N$. Interpreting $(*)$ as the reaction-diffusion problem describing the dynamics of concentration $u_1(x), \dots, u_N(x)$, $x \in \Omega$, of N reactants being subject to diffusion and reaction term, the usual assumption of nonnegativity of φ not realistic. Assumption $\varphi \geq 0$ implies that during chemical processes all substances are only produced, while, in fact, during reaction some reactants vanish or are transformed into another compounds. The realistic assumption is that if a reactant i vanishes in some area (i.e., $u_i = 0$), its amount in this area cannot decrease. This observation leads immediately to tangency (observe that if $u \in K$ with $u_i = 0$, then $T_K(u) = \{v \in \mathbb{R}^n \mid v_i \geq 0\}$) $\varphi(u) \cap T_K(u) \neq \emptyset$ meaning exactly that u_i can only increase.

Put $\mathbb{E} := L^2(\Omega, \mathbb{R}^N)$, $D(A) := H_0^1(\Omega, \mathbb{R}^N) \cap H^2(\Omega, \mathbb{R}^N)$ and define $A : D(A) \rightarrow \mathbb{E}$ by

$$(5) \quad Au := (\Delta u_1, \dots, \Delta u_N)$$

for $u = (u_1, \dots, u_N) \in D(A)$, where Δu_i denotes the usual Laplacian of a function $u_i : \Omega \rightarrow \mathbb{R}$. In view of [27, Theorem 7.2.5] A generates an analytic and resolvent compact semigroup of contractions $\{S(t)\}_{t \geq 0}$, i.e., $M = 1$ and $\omega < 0$ in Remark 3.1 (a).

Let

$$(6) \quad \mathcal{K} := \{v \in \mathbb{E} \mid v(x) \in K \text{ for a.e. } x \in \Omega\}.$$

It is immediate to see that \mathcal{K} is closed convex. In order to get (K) , i.e., to show that $J_h(\mathcal{K}) \subset \mathcal{K}$ when $h > 0$ and $h\omega < 1$ observe that

$$K = \bigcap_{y \in K} (y + T_K(y))$$

and, hence, it is sufficient to consider the case when $K = y_0 + C$, where $y_0 \in \mathbb{R}^N$ and $C \subset \mathbb{R}^N$ is a closed convex cone. Then $\mathcal{K} = u_0 + \mathcal{C}$ where $u_0(x) \equiv y_0$ on Ω and $\mathcal{C} := \{u \in \mathbb{E} \mid u(x) \in C \text{ a.e. on } \Omega\}$. Since $J_h(u_0) = u_0$, it is sufficient to show the invariance of \mathcal{C} .

Take $v \in \mathcal{C}$. Since C^∞ functions in \mathcal{C} are dense in \mathcal{C} we may assume that v is C^∞ . Let $u = J_h(v)$; by the classical regularity theory $u \in C^\infty(\text{cl } \Omega)$ and $u|_{\partial\Omega} = 0$. Let

$$p \in C^* := \{p \in \mathbb{R}^N \mid \langle y, p \rangle \geq 0 \text{ for all } y \in C\}.$$

Then $v_p := \langle p, v(\cdot) \rangle$, $u_p := \langle p, u(\cdot) \rangle$ are C^∞ , $v_p \geq 0$ on Ω and $\langle p, Au(\cdot) \rangle = \Delta u_p$. Let $x_0 \in \text{cl } \Omega$ be such that $u_p(x) \geq u_p(x_0)$ for all $x \in \text{cl } \Omega$. If $x_0 \in \partial\Omega$, then $u_p(x) \geq 0$ on Ω . If $x_0 \in \Omega$, then the second derivative $D^2 u_p(x_0)$ is nonnegative; this implies that $\Delta u_p(x_0) \geq 0$. Hence

$$u_p(x_0) = \langle p, (u - hAu)(x_0) \rangle + h \langle p, Au(x_0) \rangle = v_p(x_0) + h \Delta u_p(x_0)$$

¹See (3) with K replacing \mathcal{K} .

and again $u_p \geq 0$ on Ω . Since p was arbitrary, we gather that $u(x) \in C$ on Ω , i.e., $u \in \mathcal{C}$.

We have shown

Proposition 3.3. *If A is given by (5) and \mathcal{K} by (6), then assumptions (A) and (K) are satisfied.* \square

Let $F : [0, 1] \times \mathcal{K} \rightharpoonup \mathbb{E}$ be the *Nemytskii operator* associated with φ , i.e.

$$(7) \quad F(t, u) := \{v \in \mathbb{E} \mid v(x) \in \varphi(t, x, u(x)) \text{ for a.e. } x \in \Omega\}$$

for $t \in [0, 1], u \in \mathcal{K}$. The values of F are clearly nonempty, but *not compact* in general.

Proposition 3.4. *If φ satisfies conditions $(\varphi_1) - (\varphi_3)$, then assumption $(F_1) - (F_3)$ are satisfied. In fact F is H -usc (with respect to both variables). Assumption (φ_3) implies (F_4) . Moreover any (mild) solution to (1) is a (strong) solutions to $(*)$.*

Proof. It is easy to see (F_1) and (F_3) . Suppose F is not H -usc, i.e., there are $\varepsilon_0 > 0$, sequences $(t_n, u_n) \rightarrow (t_0, u_0)$ in $[0, 1] \times \mathbb{E}$ and $v_n \in F(t_n, u_n)$ such that

$$(8) \quad v_n \notin F(t_0, u_0) + B_{\mathbb{E}}(0, \varepsilon_0), \quad n \geq 1.$$

Up to a subsequence $(u_n)_{n \geq 1}$ converges a.e. on Ω to u_0 and there is $h \in L^2(\Omega, \mathbb{R})$ such that $|u_n(x)| \leq h(x)$ for a.e. $x \in \Omega$ and every $n \geq 0$. By (φ_3)

$$|v_n(x)| \leq \alpha(x) + c|u_n(x)| \leq \alpha(x) + ch(x) \text{ for } n \geq 0 \text{ and a.e. } x \in \Omega.$$

There is $\eta > 0$ such that for $A \subset \Omega$ with Lebesgue measure $\mu(A) < \eta$

$$(9) \quad \int_A 4(\alpha(x) + ch(x))^2 dx < \varepsilon_0^2/2.$$

For each $n \geq 0$, $H_n := \varphi(t_n, \cdot, u_n(\cdot)) : \Omega \rightharpoonup \mathbb{R}^N$ is measurable and if $w : \Omega \rightarrow \mathbb{R}^N$ is a measurable selection of H_n , then $w \in \mathbb{E}$ since, in view of (φ_3) ,

$$(10) \quad |w(x)| \leq \alpha(x) + ch(x) \text{ for a.e. } x \in \Omega.$$

By the Egorov and Lusin theorems (see [4, Th. 1] for a multivalued version of the Lusin theorem) there is a compact $\Omega_\eta \subset \Omega$ such that $\mu(\Omega \setminus \Omega_\eta) < \eta$, $u_n \rightarrow u_0$ uniformly on Ω_η , the restriction $u_0|_{\Omega_\eta} : \Omega_\eta \rightarrow \mathbb{R}^N$ is continuous and $H_0|_{\Omega_\eta} : \Omega_\eta \rightharpoonup \mathbb{R}^N$ is H -lsc.

Let $\delta := \varepsilon_0 / \sqrt{2\mu(\Omega)}$. We will show that there is n_0 such that if $n \geq n_0$ and $x \in \Omega_\eta$, then

$$H_n(x) \subset H_0(x) + B_{\mathbb{R}^N}(0, \delta).$$

Suppose to the contrary that there is a subsequence $(n_j)_{j \geq 1}$ and a sequence $(x_j)_{j \geq 1}$ in Ω_η such that

$$(11) \quad H_{n_j}(x_j) \not\subset H_0(x_j) + B_{\mathbb{R}^N}(0, \delta).$$

We can assume that $x_j \rightarrow x_0 \in \Omega_\eta$, since Ω_η is compact. The continuity of $u_0|_{\Omega_\eta}$ and the uniform convergence $u_n \rightarrow u_0$ on Ω_η imply that $u_{n_j}(x_j) \rightarrow u_0(x_0)$ and thus $(t_{n_j}, x_j, u_{n_j}(x_j)) \rightarrow (t_0, x_0, u_0(x_0))$ as $j \rightarrow \infty$. The upper semicontinuity of φ together with the H -lower semicontinuity of H_0 on Ω_η show that $H_{n_j}(x_j) \subset H_0(x_j) + B_{\mathbb{R}^N}(0, \delta)$ for sufficiently large j , which contradicts (11).

Let us fix $n \geq n_0$. For a.e. $x \in \Omega_\eta$ we have

$$(12) \quad v_n(x) \in H_n(x) \subset H_0(x) + B_{\mathbb{R}^N}(0, \delta).$$

Observe that the map $\Omega_\eta \ni x \mapsto B_{\mathbb{R}^N}(v_n(x), \delta) \cap H_0(x)$ is measurable and has nonempty values for a.e. $x \in \Omega_\eta$. By the Kuratowski–Ryll–Nardzewski theorem, there is a measurable selection $v : \Omega_\eta \rightarrow \mathbb{R}^N$, i.e. $v(x) \in B_{\mathbb{R}^N}(v_n(x), \delta) \cap H_0(x)$ for a.e. $x \in \Omega_\eta$. Clearly $v \in L^2(\Omega_\eta, \mathbb{R}^N)$ and for a.e. $x \in \Omega_\eta$, $|v_n(x) - v(x)| < \delta$. Thus

$$\int_{\Omega_\eta} |v_n(x) - v(x)|^2 dx < \delta^2 \mu(\Omega_\eta) < \varepsilon_0^2/2.$$

Take an arbitrary selection $w : \Omega \rightarrow \mathbb{R}^N$ of H_0 , i.e. $w(x) \in H_0(x)$ for a.e. $x \in \Omega$. Let χ be the indicator of Ω_η . Notice that $\chi v + (1 - \chi)w : \Omega \rightarrow \mathbb{R}^N$ is a square-integrable selection of H_0 (we identify $v : \Omega_\eta \rightarrow \mathbb{R}^N$ with the function $v : \Omega \rightarrow \mathbb{R}^N$ putting $v \equiv 0$ on $\Omega \setminus \Omega_\eta$). By (10)

$$|v_n(x) - w(x)| \leq |v_n(x)| + |w(x)| \leq 2(\alpha(x) + ch(x)) \quad \text{for a.e. } x \in \Omega \setminus \Omega_\eta.$$

Recall that $\mu(\Omega \setminus \Omega_\eta) < \eta$, hence and by (9)

$$\begin{aligned} \|v_n - \chi v + (1 - \chi)w\|^2 &= \int_{\Omega_\eta} |v_n(x) - v(x)|^2 dx + \int_{\Omega \setminus \Omega_\eta} |v_n(x) - w(x)|^2 dx \\ &< \varepsilon_0^2/2 + \int_{\Omega \setminus \Omega_\eta} 4(\alpha(x) + h(x))^2 dx < \varepsilon_0^2. \end{aligned}$$

Thus, contrary to (8), $v_n \in F(t_0, u_0) + B_{L^2(\Omega, \mathbb{R}^N)}(0, \varepsilon_0)$ for infinitely many $n \geq 1$.

In order to check (F_4) fix $t \in [0, 1]$, $u \in \mathcal{K}$ and define $G, H : \Omega \rightarrow \mathbb{R}^N$, by

$$G(x) := \varphi(t, x, u(x)), \quad H(x) := T_K \circ u(x) \quad \text{for } x \in \Omega.$$

The map $T_K(\cdot) : K \rightarrow \mathbb{R}^N$ is lsc, G is measurable; hence $\Omega \ni x \mapsto G(x) \cap H(x) \subset \mathbb{R}^N$ is measurable with nonempty values. By the Kuratowski–Ryll–Nardzewski theorem, there is a measurable $v : \Omega \rightarrow \mathbb{R}^N$ such that $v(x) \in G(x) \cap H(x)$ for a.e. $x \in \Omega$. Clearly $v \in \mathbb{E}$ and $v \in T_{\mathcal{K}}(u) \cap F(t, u)$ since in view of [3, Cor. 8.5.2] $T_{\mathcal{K}}(u) = \{v \in \mathbb{E} \mid v(x) \in T_K(u(x)) \text{ a.e. in } \Omega\}$.

The last part has been established in [14] in the unconstrained case; this proof follows immediately from a general result in [28, Proposition III.2.5]. Here the same arguments apply. \square

4. EXISTENCE AND STRUCTURE OF SOLUTIONS

In this section we assume that conditions (A) , (K) , $(F_1) - (F_4)$ hold and \mathbb{E} is a *separable* Banach space. The compactness of $\{S(t)\}_{t \geq 0}$ implies that:

Lemma 4.1. *The operator $K_0 : L^1([0, 1], \mathbb{E}) \rightarrow \mathcal{C}([0, 1], \mathbb{E})$ defined by*

$$K_0(y)(t) := \int_0^t S(t-s)y(s) ds \quad \text{for } y \in L^1([0, 1], \mathbb{E}), \quad t \in [0, 1],$$

maps integrably bounded subsets of $L^1([0, 1], \mathbb{E})$ into compact subsets of $\mathcal{C}([0, 1], \mathbb{E})$. \square

We are going to show that the set of all (mild) solutions to (1) surviving in \mathcal{K} is a compact R_δ -subset of $\mathcal{C}([0, 1], \mathbb{E})$, i.e., can be represented as the intersection of a decreasing sequence of compact absolute retracts (see also e.g. [15, p. 14] for a detailed discussion of the class of R_δ -sets). In an unconstrained case this is known (see e.g. [14] or [10]). Assumptions (K) and (F_4) imply the *viability*, but certainly do not prevent that some solution escaper from \mathcal{K} ; hence it is not clear what is the structure of solutions that stay in \mathcal{K} . Apart from the presence of constraints in (1), we deal with weakly compact convex valued and not necessarily usc perturbations, while elsewhere (see e.g. [20] or [5]) compact convex valued usc perturbations are studied.

In the proof the following characterization will be used: *If $X_0 = \bigcap_{n=1}^\infty X_n$, where $X_n \neq \emptyset$ is closed contractible, $X_n \supset X_{n+1}$ for all $n \geq 1$, and the Hausdorff measure of noncompactness $\beta(X_n) \rightarrow 0$, then X_0 is an R_δ -set.*

Theorem 4.2. *For a fixed $x_0 \in \mathcal{K}$, the set X_0 of solutions in \mathcal{K} of (1) starting at x_0 is an R_δ subset of $\mathcal{C}([0, 1], \mathcal{K})$.*

Proof. Step 1. Take a sequence $(\varepsilon_n)_{n \geq 1}$ in $(0, 1)$ such that $\varepsilon_n \searrow 0$. Since $\mathcal{K} \ni x \mapsto T_{\mathcal{K}}(x) \subset \mathbb{E}$ is lsc we can apply Theorem 2.2: for every $n \geq 1$, there is $f_n : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ such that $f_n(t, \cdot)$ is continuous for $t \in [0, 1]$ and $f_n(\cdot, x)$ is measurable for every $x \in \mathcal{K}$; $f_n(t, x) \in T_{\mathcal{K}}(x)$ and $f_n(t, x) \in F(\{t\} \times B_{\mathcal{K}}(x, \varepsilon_n)) + B_{\mathbb{E}}(0, \varepsilon_n)$ for all $t \in [0, 1], x \in \mathcal{K}$.

For each $k \geq 1$, by a version of the Scorza-Dragoni theorem (cf. [23]), there is a closed subset $\bar{I}_k \subset [0, 1]$ such that $\mu([0, 1] \setminus \bar{I}_k) \leq \min\{\varepsilon_n/2^{k-n+1}; n = 1, \dots, k\}$ and the restriction $f_k|_{\bar{I}_k \times \mathcal{K}}: \bar{I}_k \times \mathcal{K} \rightarrow \mathbb{E}$ is continuous (with respect to both variables). Let $I_n := \bigcap_{k \geq n} \bar{I}_k$, $n \geq 1$. The family $\{I_n\}$ increases, consists of compact sets and $f_n|_{I_n \times \mathcal{K}}$ is continuous. Moreover $\mu([0, 1] \setminus I_n) \leq \varepsilon_n$ and $\mu\left(\bigcup_{n \geq 1} I_n\right) = 1$.

Fix $n \geq 1$; clearly $(0, 1) \setminus I_n = \bigcup_{k \geq 1} (a_k, b_k)$. Define $\hat{f}_n: [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ by

$$\hat{f}_n(t, x) := \begin{cases} f_n(t, x) & \text{for } t \in I_n, x \in \mathcal{K} \\ \frac{b_k - t}{b_k - a_k} f_n(a_k, x) + \frac{t - a_k}{b_k - a_k} f_n(b_k, x) & \text{for } t \in [a_k, b_k], x \in \mathcal{K}. \end{cases}$$

Obviously \hat{f}_n is continuous and (conv stands for the convex hull)

$$(13) \quad \hat{f}_n(t, x) \in \text{conv} F(B_{[0,1]}(t, \varepsilon_n) \times B_{\mathcal{K}}(x, \varepsilon_n)) + B_{\mathbb{E}}(0, \varepsilon_n),$$

for $t \in [0, 1]$, $x \in \mathcal{K}$, since $b_k - a_k < \varepsilon_n$ for $k \geq 1$. If $t \in I_n$, $x \in \mathcal{K}$, then

$$(14) \quad \hat{f}_n(t, x) = f_n(t, x) \in F(\{t\} \times B_{\mathcal{K}}(x, \varepsilon_n)) + B_{\mathbb{E}}(0, \varepsilon_n).$$

For any $n \geq 1$ we find easily a continuous $\alpha_n: [0, 1] \times \mathcal{K} \rightarrow (0, \infty)$ such that if $g: [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ satisfies

$$(15) \quad g(t, x) \in \hat{f}_n(B_{[0,1]}(t, \alpha_n(t, x)) \times B_{\mathcal{K}}(x, \alpha_n(t, x))) + B_{\mathbb{E}}(0, \alpha_n(t, x)),$$

then $g(t, x) \in \hat{f}_n(t, x) + B_{\mathbb{E}}(0, \varepsilon_n)$ on $[0, 1] \times \mathcal{K}$. Applying Lemma 2.1 to \hat{f}_n , we get a locally Lipschitz $g = g_n: [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ such that

$$(16) \quad g_n(t, x) \in T_{\mathcal{K}}(x)$$

and (15) holds. Hence

$$(17) \quad g_n(t, x) \in \hat{f}_n(t, x) + B_{\mathbb{E}}(0, \varepsilon_n).$$

In view of (16), (A) and [6, Th. 7.2.] the problem $\dot{u}(t) = Au(t) + g_n(t, u(t))$, $u(0) = x_0$, admits a unique solution (mild) $\bar{u}_n \in \mathcal{C}([0, 1], \mathcal{K})$.

For any $n \geq 1$ let X_n be the set of mild solutions (in \mathcal{K}) of the problem

$$\begin{cases} \dot{u}(t) \in Au(t) + F_n(t, u(t)), \\ u(0) = x_0 \in \mathcal{K}, \end{cases}$$

where $F_n: [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ is given by

$$F_n(t, x) := \begin{cases} F(\{t\} \times B_{\mathcal{K}}(x, \varepsilon_n)) + B_{\mathbb{E}}(0, 2\varepsilon_n) & \text{for } t \in I_n, x \in \mathcal{K}, \\ \text{conv} F(B_{[0,1]}(t, \varepsilon_n) \times B_{\mathcal{K}}(x, \varepsilon_n)) + B_{\mathbb{E}}(0, 2\varepsilon_n) & \text{for } t \in [0, 1] \setminus I_n, x \in \mathcal{K}. \end{cases}$$

By (17) and (13), (14), $g_n(t, x) \in F_n(t, x)$ on $[0, 1] \times \mathcal{K}$. Therefore $X_n \neq \emptyset$ since $\bar{u}_n \in X_n$. Clearly

$$(18) \quad X_0 \subset \bigcap_{n=1}^{\infty} X_n.$$

Step 2. We shall see that given a sequence (u_n) , where $u_n \in X_n$ for $n \geq 1$, then (up to a subsequence) $u_n \rightarrow u_0 \in X_0$. To this end observe that there is $w_n \in L^1([0, 1], \mathbb{E})$ such that $w_n(t) \in F_n(t, u_n(t))$ for a.e. $t \in [0, 1]$ and $u_n(t) = S(t)x + K_0(w_n)$ for $t \in [0, 1]$. The Gronwall inequality and (F_3) imply that $\sup_{n \geq 1} \|u_n\| \leq C$ for some $C \geq 0$. Thus $\{w_n\}_{n \geq 1}$ is integrably bounded by $c(1 + C)$ and, by Lemma 4.1, $\{u_n\}_{n \geq 1}$ is relatively compact, i.e., (up to a subsequence) $u_n \rightarrow u_0 \in \mathcal{C}([0, 1], \mathcal{K})$.

Observe now that the set $\{\chi_n w_n\}_{n \geq 1}$, where χ_n stands for the indicator of I_n , is integrably bounded. Take $t \in \bigcup_{n \geq 1} I_n$, i.e., $t \in I_n$ for $n \geq \bar{N}$ for some \bar{N} . For such n

$$(19) \quad \chi_n(t) w_n(t) = w_n(t) \in F(\{t\} \times B_{\mathcal{K}}(u_n(t), \varepsilon_n)) + B_{\mathbb{E}}(0, 2\varepsilon_n),$$

i.e., $w_n(t) \in F(t, v_n) + b_n$, for some $v_n \in \mathcal{K}$, $b_n \in \mathbb{E}$ with $\|u_n(t) - v_n\| < \varepsilon_n$, $\|b_n\| < 2\varepsilon_n$. Hence

$$\|u_0(t) - v_n\| \leq \|u_0(t) - u_n(t)\| + \|u_n(t) - v_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Observe that

$$\{w_n(t) \chi_n(t)\}_{n \geq 1} \subset F\left(\{t\} \times \{v_n\}_{n \geq 1}\right) + \{b_n\}_{n \geq 1} \cup \{0\},$$

where $F\left(\{t\} \times \{v_n\}_{n \geq N}\right)$ is relatively weakly compact in view of Remark 3.1 (c). By the Diestel weak compactness criterion [12, Cor. 2.6], $\{\chi_n w_n\}_{n \geq 1}$ is relatively weakly compact in $L^1([0, 1], \mathbb{E})$, i.e., up to a subsequence $\chi_n w_n \rightharpoonup w_0 \in L^1([0, 1], \mathbb{E})$ (weakly) and, hence, $(K_0(w_n \chi_n))_{n \geq 1} \rightharpoonup K_0(w_0)$ in $\mathcal{C}([0, 1], \mathbb{E})$.

On the other hand $\|K_0((1 - \chi_n)w_n)\| \leq Rc(1 + M)\mu([0, 1] \setminus I_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$u_n - S(\cdot)x = K_0(\chi_n w_n) + K_0((1 - \chi_n)w_n) \rightharpoonup K_0(w_0).$$

This shows $u_0(t) = S(t)x + \int_0^t S(t-s)w_0(s)ds$ for $t \in [0, 1]$. In view of (19) and the ‘convergence theorem’ [2, Th. 3.2.6], $w_0(t) \in F(t, u_0(t))$ for a.e. $t \in [0, 1]$, i.e., $u_0 \in X_0$.

The assertion we have just proved together with (18) implies that X_0 is compact, $\sup_{v \in X_n} d(v; X_0) \rightarrow 0$ and, hence, the measure of noncompactness $\beta(\text{cl } X_n) \rightarrow 0$ and $X_0 = \bigcap_{n=1}^{\infty} \text{cl } X_n$.

Step 3. Now we shall show that $\text{cl } X_n$ is contractible. To see this fix $n \geq 1$ and recall the above constructed locally Lipschitz $g_n : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ being tangent to \mathcal{K} and having sublinear growth. Take $z \in [0, 1]$ and $y \in \mathcal{K}$. The problem

$$\begin{cases} \dot{u}(t) = Au(t) + g_n(t, u(t)), \\ u(z) = y, \end{cases}$$

admits a unique solution $v(\cdot; z, y) : [z, 1] \rightarrow \mathcal{K}$. The strong continuity of $\{S(t)\}_{t \geq 0}$ along with local lipschitzianity of g_n imply that $v(\cdot; z, y)$ depends continuously on $z \in [0, 1]$ and $y \in \mathcal{K}$. Precisely, given $\varepsilon > 0$, $z_0 \in [0, 1]$ and $y_0 \in \mathcal{K}$ there is $\delta > 0$ such that $\|v(t; z_0, y_0) - v(t; z, y)\| < \varepsilon$ for all $t \in [\max\{z_0, z\}, 1]$, if $|z - z_0| < \delta$, $\|y - y_0\| < \delta$.

Let us consider the homotopy $h : \text{cl } X_n \times [0, 1] \rightarrow \mathcal{C}([0, 1], \mathcal{K})$ given by

$$[h(u, z)](s) := \begin{cases} u(s) & \text{for } s \in [0, z]; \\ v(s; z, u(z)) & \text{for } s \in [z, 1] \end{cases}$$

where $u \in \text{cl } X_n$, $z \in [0, 1]$. It is easy to see that h is well-defined, continuous (comp. [7, Th. 5.1]) and $h(X_n \times [0, 1]) \subset X_n$ since g_n is the selection of F_n ; thus $h(\text{cl } X_n \times [0, 1]) \subset \text{cl } X_n$. Furthermore $h(\cdot, 0) = v(\cdot; 0, x)$ and $h(\cdot, 1) = \text{id}_{\text{cl } X_n}$ proving the contractibility of $\text{cl } X_n$. \square

4.1. c - admissible maps. Recall (see [24] and [18]) that a compact metric space S is *cell-like* if it can be represented as the intersection of a decreasing sequence of compact contractible spaces. The following conditions are equivalent (see e.g. [18]): S is cell-like; S has the shape of a point; S is an R_δ -set; S has the UV^∞ -property, i.e., if S is embedded into an ANR, then it is contractible in any of its neighborhoods.

Let X, Y be metric spaces; an usc map $\varphi : X \multimap Y$ is *cell-like* if $\varphi(x)$, $x \in X$, is cell-like. A map $\varphi : X \multimap Y$ is *c-admissible* if there is a metric space Z , a cell-like map $\psi : X \multimap Z$ and a continuous $f : Z \rightarrow Y$ such that $\varphi = f \circ \psi$. Equivalently (see [13, Section 3]) $\varphi : X \multimap Y$ is *c-admissible* if it is *represented* by a *c-admissible pair* (p, q) , i.e., $\varphi(x) = q(p^{-1}(x))$ for $x \in X$, where $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$, Γ is a metric space, p, q are continuous and p is a proper surjection with cell-like $p^{-1}(x)$, $x \in X$. Properties of a *c-admissible* φ strongly depend on a decomposition $\varphi = f \circ \psi$ or a pair (p, q) representing it. When studying *c-admissible* maps one has to take into account representing pairs (for a detailed discussion of *c-admissible* maps, related topics and some references – see [13]). In particular: if $\varphi : X \multimap Y$ is cell-like, then the *canonical* pair (p_φ, q_φ) , where the *graph* $\text{Gr}(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$, $p_\varphi : \text{Gr}(\varphi) \rightarrow X$ and $q_\varphi : \text{Gr}(\varphi) \rightarrow Y$ are projections, is *c-admissible* and represents φ . If $X \subset \mathbb{E}$, then a *c-admissible*

pair (p, q) is *compact* if $\text{cl } q(p^{-1}(B))$ is compact for any bounded $B \subset X$; $\varphi : X \multimap Y$ is *compact* if represented by a compact c -admissible pair.

After [13, Definition 3.5] we say that c -admissible pairs $X \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} Y$, $k = 0, 1$, (and set-valued maps represented by them) are *c-homotopic* (written $(p_0, q_0) \simeq (p_1, q_1)$) if there is a c -admissible pair $X \times [0, 1] \xleftarrow{p} \Gamma \xrightarrow{q} Y$ and continuous maps $j_k : \Gamma_k \rightarrow \Gamma$, $k = 0, 1$, such that the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_0} & \Gamma_0 & & \\ i_0 \downarrow & & j_0 \downarrow & \searrow q_0 & \\ X \times [0, 1] & \xleftarrow{p} & \Gamma & \xrightarrow{q} & Y, \\ i_1 \uparrow & & j_1 \uparrow & \nearrow q_1 & \\ X & \xleftarrow{p_1} & \Gamma_1 & & \end{array}$$

where $i_k(x) := (x, k)$ for $x \in X$ and $k = 0, 1$, is commutative. The pair (p, q) is called a *c-homotopy* joining (p_0, q_0) to (p_1, q_1) .

Let $\Sigma : \mathcal{K} \multimap \mathcal{C}([0, 1], \mathcal{K})$ assign to $x \in \mathcal{K}$ the set of all solutions to (1) starting at x .

Lemma 4.3. Σ is a cell-like map and maps bounded sets onto bounded ones.

Proof. The second assertion follows from the Gronwall inequality and (F_5) . In view of Theorem 4.2 we need to show that Σ is usc. Let $x_n \rightarrow x \in \mathcal{K}$ and $u_n \in \Sigma(x_n)$ for $n \geq 1$. Then $u_n = S(\cdot)x_n + K_0(w_n)$ for some $w_n \in L^1([0, 1], \mathbb{E})$ such that $w_n(t) \in F(t, u_n(t))$ for a.e. $t \in [0, 1]$. The condition (F_3) and the Gronwall inequality imply that $\{u_n\}_{n \geq 1}$ is bounded, so $\{w_n\}_{n \geq 1}$ is integrably bounded. As above (up to a subsequence) $u_n - S(\cdot)x_n \rightarrow u - S(\cdot)x$ in $\mathcal{C}([0, 1], \mathcal{K})$. Thus, again up to a subsequence $w_n \rightharpoonup w \in L^1([0, 1], \mathbb{E})$ and $w(t) \in F(t, u(t))$ for a.e. $t \in [0, 1]$. As a result, $u = S(\cdot)x + K_0(w) \in \Sigma(x)$. \square

In what follows Σ will be identified it with its canonical pair

$$(20) \quad \mathcal{K} \xleftarrow{p} \Gamma \xrightarrow{q} \mathcal{C}([0, 1], \mathcal{K}), \quad \Sigma(x) = q_\Sigma(p_\Sigma^{-1}(x)), \quad x \in \mathcal{K},$$

where $\Gamma := \{(x, u) \in \mathcal{K} \times \mathcal{C}([0, 1], \mathcal{K}) \mid u \in \Sigma(x)\}$ is the graph of Σ , p_Σ and q_Σ are the projections onto \mathcal{K} and into $\mathcal{C}([0, 1], \mathcal{K})$, respectively.

For a fixed $t \in [0, 1]$, the evaluation $e_t : \mathcal{C}([0, 1], \mathcal{K}) \rightarrow \mathcal{K}$, $e_t(u) := u(t)$ for $u \in \mathcal{C}([0, 1], \mathcal{K})$ is defined and continuous. With (1) we associate the *Poincaré t-operator* $\Sigma_t : \mathcal{K} \multimap \mathcal{K}$,

$$(21) \quad \Sigma_t := e_t \circ \Sigma, \quad \text{i.e., } \Sigma_t(x) = \{u(t) \mid u \in \Sigma(x)\}.$$

Therefore Σ_t is c -admissible (cf. [13, Rem. 3.4. (2)]); it is represented by the c -admissible pair

$$(22) \quad \mathcal{K} \xleftarrow{p_t} \Gamma \xrightarrow{q_t} \mathcal{K}, \quad \text{where } p_t := p_\Sigma, \quad q_t := e_t \circ q_\Sigma.$$

Remark 4.4. (1) The mapping $\mathcal{K} \times [0, 1] \ni (t, x) \mapsto \Sigma_t(x) \subset \mathcal{C}([0, 1], \mathcal{K})$ is c -admissible. Is it represented by the pair

$$\mathcal{K} \times [0, 1] \xleftarrow{p} \Gamma \times [0, 1] \xrightarrow{q} \mathcal{K}$$

where $p := p_\Sigma \times \text{id}_{[0, 1]}$, $q(\gamma, t) := e_t \circ q_\Sigma(\gamma)$ for $t \in [0, 1]$, $\gamma \in \Gamma$.

(2) For any numbers $0 < a \leq b \leq 1$, the restriction $[a, b] \times \mathcal{K} \ni (t, x) \mapsto \Sigma_t(x) \subset \mathcal{K}$ is completely continuous, what is a consequence of the compactness of the semigroup $\{S(t)\}_{t \geq 0}$. It is worth to emphasize that, in particular, Σ_t and the pair (p_t, q_t) are completely continuous.

Remark 4.5. A parameterized version of the above results will also be useful. Let Z be a compact metric space and let $F : Z \times [0, 1] \times \mathcal{K} \multimap \mathbb{E}$ be (product) measurable, $F(\cdot, t, \cdot)$, $t \in [0, 1]$ be H-usc and $F(z, \cdot, \cdot)$, $z \in Z$, satisfy assumptions $(F_1) - (F_4)$. Then all above results remain true, in particular: the solution map $\Sigma : Z \times \mathcal{K} \multimap \mathcal{C}([0, 1], \mathcal{K})$ is usc with R_δ -values.

4.2. Fixed point index for c - admissible maps. Given an open bounded $V \subset \mathbb{E}$, a compact c -admissible pair $\text{cl } V \xleftarrow{p} \Gamma \xrightarrow{q} \mathbb{E}$ representing it, such that $x \notin q(p^{-1}(x))$ for $x \in \partial V$, the fixed point index $\text{Ind}((p, q), V)$ is well-defined (cf. [13, Th. 4.5]). This index has the usual properties such as: the existence, the localization, the additivity and the homotopy invariance (see [13]).

It is easy to get a generalization of the above mentioned fixed point index to a constrained case in a standard way. Let $\mathcal{K} \subset \mathbb{E}$ be convex closed and let $U \subset \mathcal{K}$ be (relatively) open and bounded. Let $r : \mathbb{E} \rightarrow \mathcal{K}$ be an arbitrary retraction and $j : \mathcal{K} \hookrightarrow \mathbb{E}$ be the inclusion. Given a c -admissible compact pair $\text{cl}_{\mathcal{K}} U \xleftarrow{p} \Gamma \xrightarrow{q} \mathcal{K}$ such that $x \notin q(p^{-1}(x))$ for $x \in \partial_{\mathcal{K}} U$ ($\text{cl}_{\mathcal{K}} U$ and $\partial_{\mathcal{K}} U$ denote the closure and the boundary of U in \mathcal{K}), we let $V := r^{-1}(U) \cap B$, where B is open bounded and $B \supset U$, $\bar{\Gamma} := \{(x, \gamma) \in \text{cl } U \times \Gamma \mid r(x) = p(\gamma)\}$, $\bar{p} : \bar{\Gamma} \rightarrow \text{cl } U$ and $\bar{q} : \bar{\Gamma} \rightarrow \mathbb{E}$ by $\bar{p}(x, \gamma) := x$ and $\bar{q}(x, \gamma) := q(\gamma)$ for $(x, \gamma) \in \bar{\Gamma}$. Note that $q_r \circ p_r^{-1} = j \circ q \circ p^{-1} \circ r_U : \text{cl } U_r \rightarrow \mathbb{E}$, the pair (p_r, q_r) is compact and c -admissible and $x \notin \bar{q}(\bar{p}^{-1}(x))$ for $x \in \partial V$. Thus we are in a position to define the *constrained fixed point index* by

$$\text{Ind}_{\mathcal{K}}((p, q), U) := \text{Ind}((p_r, q_r), \text{cl } U_r).$$

It is easy to see that this definition is correct, i.e, it does not depend on the choice of r ; furthermore $\text{Ind}_{\mathcal{K}}$ has the same properties as Ind does.

Remark 4.6. (i) In particular, if two c -admissible pairs $\text{cl}_{\mathcal{K}} U \xleftarrow{p_j} \Gamma_j \xrightarrow{q_j} \mathcal{K}$, $j = 0, 1$, are c -homotopic and the c -homotopy $\text{cl}_{\mathcal{K}} U \times [0, 1] \xleftarrow{p} \Gamma \xrightarrow{q} \mathcal{K}$ is compact and such that $x \notin q(p^{-1}(x, t))$ for $x \in \partial_{\mathcal{K}} U$, $t \in [0, 1]$, then $\text{Ind}_{\mathcal{K}}((p_j, q_j), U)$, $j = 0, 1$, are defined and equal.

(ii) If a compact c -admissible pair (p, q) represents a single-valued $f : \text{cl } U \rightarrow \mathcal{K}$ and $x \neq f(x)$ for $x \in \partial U$, then it can be proved $\text{Ind}_{\mathcal{K}}((p, q), U) = \text{Ind}_{\mathcal{K}}(f, U)$, where $\text{Ind}_{\mathcal{K}}(f, U)$ stands for the fixed point index as defined in [17, §12]. In particular f is represented by the pair $\text{cl}_{\mathcal{K}} U \xleftarrow{\text{id}} \text{cl}_{\mathcal{K}} U \xrightarrow{f} \mathcal{K}$

5. THE DEGREE OF THE RIGHT HAND SIDE

We will construct a homotopy invariant (the so-called *constrained topological degree*) responsible for the existence of zeros of maps of the form $A + G$, where:

- (G₁) $G : \mathcal{K} \rightarrow \mathbb{E}$ is H-usc, has convex weakly compact values, maps bounded sets onto bounded ones and $G(x) \cap T_{\mathcal{K}}(x) \neq \emptyset$ for every $x \in \mathcal{K}$, i.e., G is tangent to \mathcal{K} ;
- (G₂) $\mathcal{K} \subset \mathbb{E}$ is convex closed; $A : D(A) \rightarrow \mathbb{E}$ satisfies (A) and (K).

Let $U \subset \mathcal{K}$ be bounded and relatively open in \mathcal{K} . We assume that

$$(23) \quad 0 \notin Ax + G(x) \quad \text{for } x \in D(A) \cap \partial U;$$

here $\partial U = \partial_{\mathcal{K}} U$ stands for the boundary of U in \mathcal{K} .

Lemma 5.1. *There is $\alpha_0 > 0$ such that if $0 < \alpha \leq \alpha_0$, then*

$$0 \notin Ax + G(B_{\mathcal{K}}(x, \alpha)) + B_{\mathbb{E}}(0, \alpha) \quad \text{for } x \in D(A) \cap \partial U.$$

Proof. Suppose to the contrary that for $n \geq 1$ there is $x_n \in D(A) \cap \partial U$, $y_n \in G(\bar{x}_n)$, where $\|x_n - \bar{x}_n\| < 1/n$ and $\xi_n \in \mathbb{E}$ with $\|\xi_n\| < 1/n$ such that

$$0 = Ax_n + y_n + \xi_n \iff x_n = J_h(x_n + h(y_n + \xi_n))$$

for fixed $h > 0, h\omega < 1$. Clearly $\{y_n\}_{n \geq 1}$ is bounded since so is $\{x_n\}$. The compactness of J_h implies that $\{x_n\}_{n \geq 1}$ is relatively compact; thus, up to a subsequence, $x_n \rightarrow x_0 \in \partial U$ and $\bar{x}_n \rightarrow x_0$. Remark 3.1 (c) and the Krein-Šmulian theorem imply that $\{y_n\}_{n \geq 1}$ is relatively weakly compact. Thus, up to a subsequence $y_n \rightharpoonup y_0$. This (see again Remark 3.1 (c)) implies that $y_0 \in G(x_0)$. Moreover $x_n = J_h(x_n + h(y_n + \xi_n)) \rightarrow J_h(x_0 + hy_0)$, since J_h is compact. Hence $x_0 = J_h(x_0 + hy_0)$, $x_0 \in D(A)$ and $0 = Ax_0 + y_0$: a contradiction. \square

Lemma 5.2. *If a continuous map $g : \mathcal{K} \rightarrow \mathbb{E}$ is tangent to \mathcal{K} , then for every $x \in \mathcal{K}$ we have*

$$\lim_{h \rightarrow 0^+, y \rightarrow x, y \in \mathcal{K}} \frac{d(J_h(y + hg(y)); \mathcal{K})}{h} = 0.$$

Proof. Take $x \in \mathcal{K}$ and $\varepsilon > 0$. The continuity and the tangency of g together with [3, Prop. 4.2.1] imply

$$\lim_{h \rightarrow 0^+, y \rightarrow x, y \in \mathcal{K}} \frac{d(y + hg(y); \mathcal{K})}{h} = 0 \quad \text{for } x \in \mathcal{K}.$$

Hence (see Remark 3.1 (a)), there is $\delta > 0$ such that if $\|y - x\| < \delta$, $0 < h < \delta$.

$$d(y + hg(y); \mathcal{K}) < \frac{\varepsilon}{2M}h \quad \text{and} \quad \frac{M}{1 - hw} < 2M.$$

Choose $k \in \mathcal{K}$ with $\|y + hg(y) - k\| < \varepsilon h (2M)^{-1}$. For $e := (k - y - hg(y))/h$, $\|e\| < \varepsilon/2M$ and $y + h(g(y) + e) = k \in \mathcal{K}$. Assumption (K) implies $J_h(y + h(g(y) + e)) \in \mathcal{K}$. Thus

$$d(J_h(y + hg(y)); \mathcal{K}) \leq \|J_h(y + hg(y)) - J_h(y + h(g(y) + e))\| \leq h\|J_h\|\|e\| < h\varepsilon$$

if $\|y - x\| < \delta$, $0 < h < \delta$. □

Let $r : \mathbb{E} \rightarrow \mathcal{K}$ be a retraction, such that $\|x - r(x)\| \leq 2d(x; \mathcal{K})$ for $x \in \mathbb{E}$; such retractions exist.

Lemma 5.3. *Assume that $g : K \rightarrow \mathbb{E}$ is continuous and tangent α -approximation of G , i.e., $g(x) \in G(B_{\mathcal{K}}(x, \alpha)) + B_{\mathbb{E}}(0, \alpha)$ for $x \in \mathcal{K}$, where $0 < \alpha \leq \alpha_0$ (see Lemma 5.1). Then there is $h_0 > 0$, $h_0\omega < 1$ such that for $h \in (0, h_0]$*

$$x \neq r \circ J_h(x + hg(x)) \quad \text{for } x \in \partial U.$$

Proof. If not, then for each $n \geq 1$ there is $x_n \in \partial U$ such that $x_n = r \circ J_{h_n}(x_n + h_n g(x_n))$, where $0 < h_n < 1/n$ and $h_n\omega < 1$. Denoting $u_n := J_{h_n}(x_n + h_n g(x_n)) \in D(A)$ we have $h_n^{-1}d(u_n; \mathcal{K}) \rightarrow 0$ in view of Lemma 5.2 and

$$u_n - r(u_n) = u_n - x_n = h_n(Au_n + g(x_n)).$$

Hence

$$(24) \quad \|Au_n + g(x_n)\| = \frac{1}{h_n}\|u_n - r(u_n)\| \leq \frac{2}{h_n}d(u_n; \mathcal{K}) \rightarrow 0.$$

Thus $\{Au_n\}_{n \geq 1}$ is bounded since so is $\{g(x_n)\}_{n \geq 1}$. Note that $\|u_n\| \leq \|J_{h_n}\|\|x_n + h_n g(x_n)\| \leq R$ for some $R > 0$. Fix $h > 0$, $h\omega < 1$. The compactness of J_h and $u_n = J_h(u_n - hAu_n)$ implies that, up to a subsequence, $u_n \rightarrow x_0 \in \mathbb{E}$. Since $d(u_n; \mathcal{K}) \rightarrow 0$, we infer that $x_0 \in \mathcal{K}$ and $x_n = r(u_n) \rightarrow r(x_0) = x_0 \in \partial U$. In view of (24) $Au_n \rightarrow -g(x_0)$ and since A is closed we have $x_0 \in D(A)$ and $Ax_0 = -g(x_0)$. As a result $x_0 \in D(A) \cap \partial U$ and $0 = Ax_0 + g(x_0)$: a contradiction to Lemma 5.1. □

By Lemma 2.1 there is a locally Lipschitz $g : \mathcal{K} \rightarrow \mathbb{E}$ tangent to \mathcal{K} being an α -approximation of G . Let $h \in (0, h_0]$ (h_0 is taken from Lemma 5.3) and consider $f : \text{cl } U \rightarrow \mathcal{K}$ defined by

$$f(x) := r \circ J_h(x + hg(x)) \quad \text{for } x \in \text{cl } U.$$

Obviously, f is compact and by Lemma 5.3, $x \neq f(x)$ for $x \in \partial U$. Thus, the fixed point index in ANRs $\text{Ind}_{\mathcal{K}}(f, U)$ is well-defined (see [17, §12])

Lemma 5.4. *The number $\text{Ind}_{\mathcal{K}}(f, U)$ does not depend on the choice of a sufficiently close approximation g , a retraction r and sufficiently small $h > 0$.*

Proof. Take two retraction $r_0, r_1 : \mathbb{E} \rightarrow \mathcal{K}$ such that $\|x - r_i(x)\| \leq 2d(x; \mathcal{K})$, $x \in \mathbb{E}$, $i = 0, 1$, and two locally Lipschitz α -approximations $g_0, g_1 : \mathcal{K} \rightarrow \mathbb{E}$ of G tangent to \mathcal{K} , where $0 < \alpha \leq \alpha_0$. If Repeating arguments from Lemmata 5.1 and 5.3 we find a sufficiently small $\alpha > 0$ and $h \leq h_0$ such that for any $t \in [0, 1]$

$$x \neq r_t \circ f_t(x) := J_h(x + hg_t(x)) \text{ on } \partial U,$$

where $r_t := (1-t)r_0 + tr_1$ and $g_t = (1-t)g_0 + tg_1$. Thus $\partial U \times [0, 1] \ni (x, t) \mapsto f_t(x)$ provides a (compact) homotopy joining f_0 to f_1 showing that $\text{Ind}_{\mathcal{K}}(f_0, U) = \text{Ind}_{\mathcal{K}}(f_1, U)$. The independence of $\text{Ind}_{\mathcal{K}}(f, U)$ follows easily from the resolvent identity

$$J_b = J_a \left[\frac{a}{b} I + \frac{b-a}{b} J_b \right],$$

being valid for any $a, b > 0$ with $a\omega, b\omega < 1$ and again the homotopy invariance of the fixed point index. \square

Thus, we are in a position to define the degree $\deg_{\mathcal{K}}$ by

$$(25) \quad \deg_{\mathcal{K}}(A + G, U) := \lim_{h \rightarrow 0^+} \text{Ind}_{\mathcal{K}}(r \circ J_h(I + hg), U)$$

where $g : \mathcal{K} \rightarrow \mathbb{E}$ is a tangent and sufficiently close locally Lipschitz approximation of G .

Proposition 5.5. *The degree $\deg_{\mathcal{K}}$ has the following basic properties:*

- (1) (Existence) *If $\deg_{\mathcal{K}}(A + G, U) \neq 0$, then there is $x \in D(A) \cap U$ such that $0 \in Ax + G(x)$;*
- (2) (Additivity) *If $U_1, U_2 \subset U$ are disjoint open in \mathcal{K} and $0 \notin Ax + G(x)$ for $x \in D(A) \cap [\text{cl } U \setminus (U_1 \cup U_2)]$, then*

$$\deg_{\mathcal{K}}(A + G, U) = \deg_{\mathcal{K}}(A + G, U_1) + \deg_{\mathcal{K}}(A + G, U_2).$$

- (3) (Homotopy invariance) *If $H : [0, 1] \times \text{cl } U \rightarrow \mathbb{E}$ is H -usc with convex weakly compact values, maps bounded sets onto bounded ones and is tangent to \mathcal{K} , i.e., $H(t, x) \cap T_{\mathcal{K}}(x) \neq \emptyset$, $t \in [0, 1]$, $x \in \partial U$, and such that $0 \notin Ax + H(t, x)$ for $t \in [0, 1]$, $x \in \partial U$, then*

$$\deg_{\mathcal{K}}(A + H(0, \cdot), U) = \deg_{\mathcal{K}}(H(1, \cdot), U).$$

Proof. Suppose $0 \notin Ax + G(x)$ for $x \in \text{cl } U \cap D(A)$. Arguing as in Lemmata 5.1 and 5.3 we find $0 < \alpha_1 \leq \alpha_0$ and $0 < h_1 \leq h_0$ such that for any $0 < \alpha \leq \alpha_1$ and any locally Lipschitz and tangent α -approximation $g : \mathcal{K} \rightarrow \mathbb{E}$, $x \neq r \circ J_h(x + hg(x))$ for $x \in \text{cl } U$, where $0 < h \leq h_1$. This shows that $\deg_{\mathcal{K}}(A + G, U) = 0$.

The remaining assertions are standard and left to the reader. \square

6. THE KRASNOSEL'SKII TYPE FORMULA

In this section we will prove the following counterpart of the classical Krasnoselskii formula by establishing a formula relating the constrained degree of the operator $A + F(0, \cdot)$ in the right-hand side of (1) and the fixed point index of the Poincaré operator Σ_t (with sufficiently small $t > 0$) associated to (1); see (21), (22).

Theorem 6.1. *Assume that operator $A : D(A) \rightarrow \mathbb{E}$, where \mathbb{E} is a separable Hilbert space, and \mathcal{K} satisfy hypotheses (A), (K) and, additionally let $\|S(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$ and all $t \geq 0$. Let $F : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ satisfy conditions (F_1) , (F_3) and $d(F_4)$ and, instead of (F_2) , we assume that*

$$(F) \quad F : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E} \text{ is } H\text{-usc.}$$

Let $U \subset \mathcal{K}$ be bounded relatively open in \mathcal{K} and $0 \notin Ax + F(0, x)$ for $x \in \partial U \cap D(A)$. There is $t_0 \in (0, 1]$ such that if $t \in (0, t_0]$, then $\text{Ind}_{\mathcal{K}}((p_t, q_t), U)$ is well-defined and equal to $\deg_{\mathcal{K}}(A + F(0, \cdot), U)$.

Observe that $F(0, \cdot)$ satisfies (G_1) and (G_2) ; hence $\deg_{\mathcal{K}}(A + F(0, \cdot), U)$ is well-defined

The proof of Theorem 6.1 will be presented in a series of steps and auxiliary lemmata.

Step 1. Define $\widehat{F} : [0, 1] \times \mathcal{K} \multimap \mathbb{E}$ by the formula

$$\widehat{F}(t, x) := \overline{\text{conv}}[F([0, t], x)] \text{ for } t \in [0, 1], x \in \mathcal{K} \text{ (}^2\text{)}.$$

Lemma 6.2. \widehat{F} has convex weakly compact values, is H-usc, has sublinear growth and is tangent to \mathcal{K} .

Proof. It is sufficient to show $[0, 1] \times \mathcal{K} \ni (t, x) \mapsto F([0, t], x) \subset \mathbb{E}$ is H-usc, for the H-upper semicontinuity and other properties of \widehat{F} follow rather easily by standard arguments. Take $t_0 \in [0, 1]$, $x_0 \in \mathcal{K}$ and $\varepsilon > 0$. For some $\delta_0 > 0$

$$(26) \quad F(t, x_0) \subset F(t_0, x_0) + B_{\mathbb{E}}(0, \varepsilon/2)$$

if $t \in (t_0 - \delta_0, t_0 + \delta_0) \cap [0, 1]$. For every $t \in [0, t_0 + \delta_0/2]$ there is $\delta(t) = \delta(t, x_0) > 0$ such that

$$F(s, x) \subset F(t, x_0) + B_{\mathbb{E}}(0, \varepsilon/2),$$

provided $s \in (t - \delta(t), t + \delta(t))$, $x \in B_{\mathcal{K}}(x_0, \delta(t))$. Let $\{(t_i - \delta(t_i), t_i + \delta(t_i))\}_{i=1, \dots, k}$ be a finite sub-cover of an open cover $\{(t - \delta(t), t + \delta(t))\}_{t \in [0, t_0 + \delta_0/2]}$ of $[0, t_0 + \delta_0/2]$. Put $\delta := \min\{\delta_0/2, \delta(t_1), \dots, \delta(t_k)\}$.

Choose $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$, $x \in B_{\mathcal{K}}(x_0, \delta)$ and let $y \in F([0, t] \times \{x\})$, i.e., $y \in F(s, x)$ for some $s \in [0, t]$. There is t_i such that $s \in (t_i - \delta(t_i), t_i + \delta(t_i))$. The inclusion $B_{\mathcal{K}}(x_0, \delta) \subset B_{\mathcal{K}}(x_0, \delta(t_i))$ implies

$$y \in F(s, x) \subset F(t_i, x_0) + B_{\mathbb{E}}(0, \varepsilon/2).$$

If $t_i \leq t_0$, then

$$y \in F(t_i, x_0) + B_{\mathbb{E}}(0, \varepsilon/2) \subset F([0, t_0], x_0) + B_{\mathbb{E}}(0, \varepsilon),$$

while if $t_i > t_0$, $t_i \in [0, t_0 + \delta/2]$, then $t_i - t_0 \leq \delta_0/2$, so by (26)

$$y \in F(t_i, x_0) + B_{\mathbb{E}}(0, \varepsilon/2) \subset F(t_0, x_0) + B_{\mathbb{E}}(0, \varepsilon) \subset F([0, t_0], x_0) + B_{\mathbb{E}}(0, \varepsilon),$$

i.e., $F([0, t], x) \subset F([0, t_0], x_0) + B_{\mathbb{E}}(0, \varepsilon)$ if $t \in [0, 1]$, $|t - t_0|, \delta$ and $x \in \mathcal{K}$, $\|x - x_0\| < \delta$. \square

Using the same methods as in Lemma 5.1 we get:

Lemma 6.3. There are $\alpha > 0$, $T > 0$ such that $0 \notin Ax + \widehat{F}(T, B_{\mathcal{K}}(x, \alpha)) + B_{\mathbb{E}}(0, \alpha)$ for $x \in \partial U$. \square

Step 2. By Lemma 2.1, there is locally Lipschitz $f : \mathcal{K} \rightarrow \mathbb{E}$ being tangent to \mathcal{K} and an α -approximation of $F(0, \cdot)$. Arguing as in Lemma 5.3 we find $h_0 > 0$, $h_0\omega < 1$ such that

$$(27) \quad x \neq r \circ J_h(x + hf(x)) \text{ for } x \in \partial U \text{ for } h \in (0, h_0].$$

Observe that, by definition (see (25)),

$$(28) \quad \deg_{\mathcal{K}}(A + F(0, \cdot), U) = \text{Ind}_{\mathcal{K}}(r \circ J_h(I + hf), U),$$

where $r : \mathbb{E} \rightarrow \mathcal{K}$ is a retraction. In what follows let r be a *metric retraction*, i.e., $\|x - r(x)\| = d(x, \mathcal{K})$ for any $x \in \mathbb{E}$.

Define the auxiliary set-valued map $G : [0, 1] \times \mathcal{K} \multimap \mathbb{E}$ by the formula

$$G(z, x) := (1 - z)f(x) + z\widehat{F}(T, x) \quad z \in [0, 1], x \in \mathcal{K}.$$

Obviously, G is H-usc, tangent to \mathcal{K} , has sublinear growth and convex weakly compact values. By Theorem 4.2, the solution set of the below problem is R_{δ} :

$$(29) \quad \begin{cases} \dot{u} = Au + G(z, x), & u \in \mathcal{K}, z \in [0, 1] \\ u(0) = x \in \text{cl } U. \end{cases}$$

²Here $F([0, t], x) := F([0, t] \times \{x\})$.

Lemma 6.4. *There is $t_0 \in (0, T]$ such that for every $t \in (0, t_0]$ no solution u of (29) starting at $x \in \partial U$ is such that $u(t) = x$.*

Proof. Suppose to the contrary that for each integer $n \geq n_0$, where $n_0^{-1} < T$ there are $x_n \in \partial U$, $t_n \in (0, n^{-1}]$, $z_n \in [0, 1]$ and the solution $u_n : [0, t_n] \rightarrow \mathcal{K}$ of (29) such that $u_n(0) = x_n = u_n(t_n)$. Then there is $w_n \in L^1([0, t_n], \mathbb{E})$ such that $w_n(s) \in G(z_n, u_n(s))$ for a.e. $s \in [0, t_n]$ and

$$(30) \quad u_n(t) = S(t)x_n + \int_0^t S(t-s)w_n(s)ds, \quad t \in [0, t_n].$$

Extending periodically, we may assume that u_n and w_n are defined on $[0, T]$, i.e. $u_n \in \mathcal{C}([0, T], \mathcal{K})$, $w_n \in L^1([0, T], \mathbb{E})$. The semigroup property ensures that formula (30) is valid for every $t \in [0, T]$ and $w_n \in G(z_n, u_n(s))$ for a.e. $s \in [0, T]$. Thus u_n is a solution on $[0, T]$ of (29).

The growth condition and Gronwall's inequality imply that $\{u_n\}_{n \geq 1}$ is bounded. Therefore $\{w_n\}_{n \geq 1}$ being a.e. bounded by a constant is weakly relatively compact in $L^1([0, T], \mathbb{E})$ (cf. [12, Cor. 2.6]). Passing to a subsequence we may assume that $w_n \rightharpoonup w_0 \in L^1([0, T], \mathbb{E})$ and $z_n \rightarrow z_0 \in [0, 1]$.

To prove that $\{u_n\}_{n \geq 1}$ is relatively compact it is enough to show that so is $\{x_n\}_{n \geq 1}$ (cf. Lemma 4.1). Take $T_0 \in (0, T)$ and put $k_n := ([T_0/t_n] + 1)$. Then $r_n := k_n t_n - T_0 \rightarrow 0$ and $u_n(k_n t_n) = x_n$ for large n . So for sufficiently large $n \geq 1$: $T_0 + r_n < T$ and

$$x_n = u_n(k_n t_n) = S(T_0 + r_n)x_n + \int_0^{T_0 + r_n} S(T_0 + r_n - s)w_n(s)ds.$$

The compactness of the semigroup yields that $\{x_n\}_{n \geq 1}$ is relatively compact and so $x_n \rightarrow x_0 \in \partial U$.

Thus $u_n \rightarrow u_0 \in \mathcal{C}([0, T], \mathcal{K})$, and by the uniform equicontinuity of $\{u_n\}_{n \geq 1}$

$$\|u_0(t) - x_0\| \leq \|u_0 - u_n\| + \|u_n(t) - u_n([t/t_n]t_n)\| + \|x_n - x_0\| \rightarrow 0,$$

hence $u(t) = x_0$ for $t \in [0, T]$. Therefore

$$(31) \quad x_0 = S(t)x_0 + \int_0^t S(t-s)w_0(s)dz$$

and $w_0(s) \in G(z_0, x_0)$ for a.e. $s \in [0, T]$. Since $[0, T] \ni t \mapsto \int_0^t w_0(s)ds$ is a.e. differentiable, take $\xi \in (0, T)$ such that $w_0(\xi) \in G(z_0, x_0)$ and $\frac{d}{dt}\Big|_{t=\xi} \int_0^t w_0(s)ds = w_0(\xi)$. By (31) for small $\eta > 0$; $x_0 = S(\eta)x_0 + \int_\xi^{\xi+\eta} S(\xi+\eta-s)w_0(s)ds$.

We show that

$$(32) \quad \frac{1}{\eta} \int_\xi^{\xi+\eta} (w_0(s) - S(\xi+\eta-s)w_0(s))ds \rightarrow 0 \quad \text{as } \eta \rightarrow 0^+.$$

Take $p \in \mathbb{E}^*$ and $\varepsilon > 0$. Then

$$\left\langle \frac{1}{\eta} \int_\xi^{\xi+\eta} (w_0(s) - S(\xi+\eta-s)w_0(s))ds, p \right\rangle = \frac{1}{\eta} \int_\xi^{\xi+\eta} \langle w_0(s), p - S^*(\xi+\eta-s)p \rangle ds$$

and since \mathbb{E} is the Hilbert space the dual semigroup $\{S^*(t)\}_{t \geq 0}$ is strongly continuous. Thus there is $\delta > 0$ such that $\|S^*(t)p - p\| < \varepsilon/M$ if $0 \leq t < \delta$ where $M := \sup_{y \in G(z_0, x_0)} \|y\|$. If $0 < \eta < \delta$, then for $\xi \leq s \leq \xi + \eta$ we have $\|S^*(\xi + \eta - s)p - p\| < \varepsilon/M$ so

$$|\langle w_0(s), p - S^*(\xi + \eta - s)p \rangle| < \varepsilon \quad \text{for a.e. } s \in [\xi, \xi + \eta],$$

what proves (32). As a result

$$\frac{S(\eta)x_0 - x_0}{\eta} = \frac{1}{\eta} \int_\xi^{\xi+\eta} (w_0(s) - S(\xi+\eta-s)w_0(s))ds - \frac{1}{\eta} \int_\xi^{\xi+\eta} w_0(s)ds \rightarrow -w_0(\xi).$$

In view of [27, Th. 2.1.3] $x_0 \in D(A) \cap \partial U$ and $Ax_0 = -w_0(\xi)$. Hence $0 = Ax_0 + w_0(\xi) \in Ax_0 + G(z_0, x_0)$ what contradicts Lemma 6.3, since $G(z_0, x_0) \subset \text{conv } F_T(B_{\mathcal{K}}(x_0, \alpha)) + B_{\mathbb{E}}(0, \alpha)$. \square

Step 3. Recall now the solution operator $\Sigma : \mathcal{K} \rightarrow \mathcal{C}([0, 1], \mathcal{K})$ (see (20)) and the t -Poincaré operator $\Sigma_t : \mathcal{K} \rightarrow \mathcal{K}$ associated with to (1) and consider their restrictions to $\text{cl } U$. By a slight abuse of notation, we will still denote these restrictions by the same symbols, i.e. $\Sigma : \text{cl } U \rightarrow \mathcal{C}([0, 1], \mathcal{K})$ is represented by the c -admissible pair

$$\text{cl } U \xleftarrow{p_\Sigma} \Gamma \xrightarrow{q_\Sigma} \mathcal{C}([0, 1], \mathcal{K})$$

with p_Σ , q_Σ and $\Gamma = \{(x, u) \in \text{cl } U \times \mathcal{C}([0, 1], \mathcal{K}) \mid u \in \Sigma(x)\}$ having the same sense as in (20), while $\Sigma_t : \text{cl } U \rightarrow \mathcal{K}$ is represented by

$$\text{cl } U \xleftarrow{p_t} \Gamma \xrightarrow{q_t} \mathcal{K}$$

with $p_t := p_\Sigma$, $q_t := e_t \circ q_\Sigma$.

Taking into account (28) and Remark 4.6 we are to show that for sufficiently small $t > 0$, $h > 0$, the c -admissible pairs (p_t, q_t) and $(\text{id}, r \circ J_h(I + hf))$, where id stand for the identity on $\text{cl } U$, are c -homotopic via a compact c -homotopy without fixed points on the boundary ∂U . This will be done in several stages.

For any $x \in \mathcal{K}$, the problem

$$(33) \quad \begin{cases} \dot{u} = Au + f(u) & \text{for } t \in I, \\ u(0) = x \end{cases}$$

possesses the unique solution $P(x)$; the map $P : \text{cl } U \rightarrow \mathcal{C}([0, 1], \mathcal{K})$ is continuous. For $t \in [0, 1]$, the Poincaré t -operator $P_t : \text{cl } U \rightarrow \mathcal{K}$ associated to (33), i.e., given by $P_t(x) := P(x)(t)$ for $x \in \text{cl } U$, is compact.

Let us consider the Poincaré operator $\Phi : [0, 1] \times \text{cl } U \rightarrow \mathcal{C}([0, 1], \mathcal{K})$ associated with the problem

$$(34) \quad \dot{u} \in Au + (1 - z)f(u) + zF(t, u), \text{ for } t \in [0, 1], z \in [0, 1], u \in \mathcal{K}.$$

It is clear that Φ is cell-like (cf. Remark 4.5). Fix $t \in [0, 1]$ and consider the Poincaré t -operator $\Phi_t : [0, 1] \times \text{cl } U \rightarrow \mathcal{K}$ defined by

$$\Phi_t(z, x) := \{u(t) \in \mathcal{K} \mid u \in \Phi(z, x)\}.$$

As before Φ_t is compact and c -admissible (cf. 4.4). If $u \in \Phi(z, x)$ for some $z \in [0, 1]$, $x \in \mathcal{K}$ then $u|_{[0, t_0]}$ is also the solution of the problem (29) on the segment $[0, t_0]$, since $F(t, y) \subset \widehat{F}(T, y)$ for $t \in [0, T]$, $y \in \mathcal{K}$. Thus, by Lemma 6.4

$$x \notin \Phi_t(z, x) \quad \text{for } t \in [0, t_0], x \in \partial U, z \in [0, 1].$$

Clearly $\Phi(1, \cdot) = \Sigma$ and $\Phi(0, \cdot) = P$ (so $\Phi_t(1, \cdot) = \Sigma_t$ and $\Phi_t(0, \cdot) = P_t$). Therefore, the canonical pair (p_Φ, q_Φ) representing Φ is the c -homotopy joining (p_Σ, q_Σ) to the canonical pair representing P . Therefore the pair $(p_\Phi, e_t \circ q_\Phi)$ representing Φ_t is a c -homotopy joining (p_t, q_t) to $(\text{id}_{\text{cl } U}, P_t)$. Hence:

Lemma 6.5. *If $t \in (0, t_0]$ (t_0 is given by Lemma 6.4) then the pairs (p_t, q_t) and $(\text{id}_{\text{cl } U}, P_t)$ are c -homotopic via the compact c -homotopy without fixed points on ∂U \square .*

Proposition 6.6. *There are $0 < t_1 \leq t_0$ and $0 < h_1 \leq h_0$ such that for $t \in (0, t_1]$, $h \in (0, h_1]$ maps P_t and $g := r \circ J_h(I + hf)$ (see (28)) are homotopic via a compact homotopy without fixed points on ∂U .*

Proof. **Claim 1.** For sufficiently small $t > 0$ and $h > 0$ the Poincaré t -operators associated with (33) and the problem

$$(P_{J_h}) \quad \begin{cases} \dot{u} = -u + g(u), \\ u(0) = x \end{cases}$$

are homotopic via a condensing (with respect to the Hausdorff measure of noncompactness) homotopy without fixed points on ∂U .

Fix $h \in (0, h_0]$ and consider a parameterized problem

$$(35) \quad \dot{u} = zA + g_z(u), \text{ for } z \in [0, 1], u \in \mathcal{K}$$

where $g_z : \mathcal{K} \rightarrow \mathbb{E}$ is defined by

$$g_z(x) := zf(x) + (1-z)(-x + g(x)) \quad \text{for } z \in [0, 1], x \in \mathcal{K}.$$

Clearly, for each $z \in [0, 1]$, g_z is locally Lipschitz, since so are f and r . Moreover, for any $x \in \mathcal{K}$, $f(x) \in T_{\mathcal{K}}(x)$ and

$$g(x) = -x + r \circ J_h(x + hf(x)) \in \mathcal{K} - x \subset T_{\mathcal{K}}(x) \quad \text{for } x \in \mathcal{K}$$

and, hence, $g_z(x) \in T_{\mathcal{K}}(x)$ for $x \in \mathcal{K}$. It is easy to see that g_z has sublinear growth and the semigroup $\{S(zt)\}_{t \geq 0}$ generated by the operator zA leaves the set \mathcal{K} invariant. Thus, for any $z \in [0, 1]$, $x \in \mathcal{K}$, the problem (35) along with the initial condition $u(0) = x$ has a unique mild solution $\Theta(x, z) : [0, 1] \rightarrow \mathcal{K}$. Obviously $\Theta(x, 0)$ is the solution to (P_{J_h}) while $\Theta(x, 1)$ is the solution to $(P_{A,f})$.

To see that, for some small $t > 0$, the map

$$\text{cl } U \times [0, 1] \ni (x, z) \mapsto \Theta_t(x, z) := \Theta(x, z)(t) \in \mathcal{K}$$

is the required homotopy joining the Poincaré t -operators of (P_{J_h}) and $(P_{A,f})$ we need to study a different form of (35). Namely consider the following family $\{A_z : D(A) \rightarrow \mathbb{E}\}_{z \in [0, 1]}$ of operators defined by

$$A_z := \left(z - 1 - \frac{z}{h}\right)I + zA \quad \text{for } z \in [0, 1]$$

and let $f_z : \mathcal{K} \rightarrow \mathbb{E}$ be given by the formula

$$f_z(x) := \left(\frac{z}{h}I + (1-z)r \circ J_h\right)(x + hf(x)), \quad \text{for } h \in (0, h_0], z \in [0, 1], x \in \mathcal{K}.$$

A straightforward calculation shows that for $z \in [0, 1]$ and $x \in \mathcal{K}$ $A_z x + f_z(x) = zAx + g_z(x)$. Hence and by the use of the formula [27, Chapter 3.1. (1.2)] and the Fubini theorem we gather that $\Theta(x, z)$ is also the unique solution to the problem

$$(36) \quad \begin{cases} \dot{u}(t) = A_z u(t) + f_z(u(t)), \\ u(0) = x. \end{cases}$$

By [11, Theorem 4.5], the operator Θ_t is continuous and condensing with respect to the Hausdorff measure of noncompactness; moreover there is $t'_1 > 0$ such that if $t \in (0, t'_1]$, then

$$(37) \quad \Theta_t(x, z) \neq x \quad \text{for } z \in [0, 1], x \in \partial U.$$

We have just shown that if $0 < t < t'_1$, then the Poincaré t -operator P_t is homotopic to the Poincaré t -operator $\Theta_t(\cdot, 0)$ associated to (P_{J_h}) via a condensing homotopy without fixed points on $[0, 1] \times \partial U$.

Claim 2. For sufficiently small $t > 0$ and sufficiently small $h > 0$, the Poincaré t -operator $\Theta_t(\cdot, 0)$ associated with (P_{J_h})

$$\dot{u} = -u + g(u) \quad \text{for } u \in \mathcal{K}$$

is homotopic to g via condensing homotopy without fixed points on ∂U .

Indeed, for a fixed t and for $x \in \text{cl } U$, $z \in [0, 1]$ let

$$\widehat{\Psi}_t(z, x) := \begin{cases} \left(1 - \frac{1}{z(t+z-zt)}\right)x + \frac{1}{z(t+z-zt)}\Theta_{zt}(x, 0) & \text{for } z \in (0, 1], \\ g(x) & \text{for } z = 0, \end{cases}$$

As in [11, Prop. 4.3] one shows that $\widehat{\Psi}_t$ is continuous and there is $t_1 \in (0, 1)$, $t_1 < t'_1$ such that $\widehat{\Psi}_t$ is condensing and $\widehat{\Psi}_t(z, x) \neq x$ for $z \in [0, 1]$, $x \in \partial U$ provided for $t \in (0, t_1]$.

Take $0 < t \leq t_1$ and let $\Psi_t := r \circ \widehat{\Psi}_t : [0, 1] \times \text{cl } U \rightarrow \mathcal{K}$. Then Ψ_t is continuous and condensing as the superposition of $\widehat{\Psi}_t$ with the nonexpansive metric projection r . We shall make of the following general observation.

Lemma 6.7. *If $x \in \mathcal{K}$, then $y \in x + \bigcup_{h>0} h(\mathcal{K} - x)$ and $r(y) = x$ if and only if $y = x$.*

Proof. There are $h_0 > 0$ and $k_0 \in \mathcal{K}$ such that $y = x + h_0(k_0 - x)$. The so-called variational character-ization of r (see e.g. [8, Th. 5.2]) yields that for all $k \in \mathcal{K}$, $\langle k - r(y), y - r(y) \rangle = \langle k - x, y - x \rangle \leq 0$. Hence $\langle k - x, (x + h_0(k_0 - x)) - x \rangle = h_0 \langle k - x, k_0 - x \rangle \leq 0$ for every $k \in \mathcal{K}$. Thus, $k_0 = x$ and $y = x + h_0(k_0 - x) = x$. \square

Observe that $\Psi_t(x, 0) = r \circ g(x) = r \circ J_h(x + hf(x))$ and, in view of (27), $\Psi_t(x, 0) \neq x$ for $x \in \partial U$. If $z \in (0, 1]$, then for $x \in \text{cl } U$

$$\widehat{\Psi}_t(x, z) = x + \frac{1}{z(t + z - zt)} (\Theta_{zt}(x, 0) - x) \in x + \bigcup_{h \geq 0} h(\mathcal{K} - x).$$

For such z and x , by Lemma 6.7, $x = \Psi_t(s, x) = r \circ \widehat{\Psi}_t(s, x)$ if and only if $x = \widehat{\Psi}_t(s, x)$. Therefore, in view of [11, Prop. 4.3., Claim 2.], $\Psi_t(s, x) \neq x$ for $s \in (0, 1], x \in \partial U$. As a result: $\Psi_t(x, z) \neq x$ for $z \in [0, 1], x \in \partial U$ provided $t \in (0, t_1]$. Finally, in order to obtain a compact homotopy joining g to $\Theta(\cdot, 0)$ we will rely on the following result.

Lemma 6.8. [1, Th. 3.1.4.], [1, Def. 3.1.7.] *Let $X \subset \mathcal{K}$ be bounded closed and $f_0, f_1 : X \rightarrow \mathcal{K}$ be compact maps. If $h : [0, 1] \times X \rightarrow \mathcal{K}$ is a condensing homotopy joining f_0 to f_1 , then there is the compact homotopy $H : [0, 1] \times X \rightarrow \mathcal{K}$ joining f_0 to f_1 having the same fixed points as h does.* \square

This establishes Proposition 6.6, since Lemma 6.8 produces a compact homotopy out of Ψ_t (recall that g and P_t are compact). \square

To sum up, we proved that for sufficiently small $t > 0$ and $h > 0$:

- (1) the c -admissible pair (p_t, q_t) is c -homotopic to the pair $(\text{id}_{\text{cl } U}, P_t)$ via the compact c -homotopy without fixed point on $[0, 1] \times \partial U$ (cf. 6.5);
- (2) the Poincaré t -operator $P_t : \text{cl } U \rightarrow \mathcal{K}$ is homotopic to $r \circ J_h(I + hf) : \text{cl } U \rightarrow \mathcal{K}$ via the compact homotopy without fixed points on ∂U (cf. Corollary ??).

Thus, in view of 4.6 we have

$$\text{Ind}_{\mathcal{K}}((p_t, q_t), U) = \text{Ind}_{\mathcal{K}}((\text{id}_{\text{cl } U}, P_t), U) = \text{Ind}_{\mathcal{K}}(P_t, U) = \deg_{\mathcal{K}}(A + F(0, \cdot), U).$$

This concludes the proof of Theorem 6.1. \square

Let us finally formulate a direct single-valued counterpart of this result being a direct generalization of [11].

Corollary 6.9. *Assume that A and U are the same as in Theorem 6.1. Additionally, let $f : [0, 1] \times \mathcal{K} \rightarrow \mathbb{E}$ be tangent to \mathcal{K} locally Lipschitz function with sublinear growth. If $0 \neq Ax + f(0, x)$, $x \in \partial U$, then there is $t_0 \in (0, 1]$ such that for every $t \in (0, t_0]$*

$$\text{Ind}_{\mathcal{K}}(P_t, U) = \deg_{\mathcal{K}}(A + f(0, \cdot), U),$$

where $P_t : \text{cl } U \rightarrow \mathcal{K}$ is the Poincaré t -operator associated with the problem $\dot{u}(t) = Au(t) + f(t, u)$.

REFERENCES

- [1] AKHMEROV R. R., KAMENSKII M. I., POTAPOVA A. S., RODKINA A. E., SADOVSKII B. N., *Measures of Noncompactness And Condensing Operators*, Basel–Berlin–Boston, Birkhäuser Verlag (1992).
- [2] AUBIN J-P., EKELAND I., *Applied Nonlinear Analysis*, Wiley, New York (1986).
- [3] AUBIN J-P., FRANKOWSKA H., *Set-valued Analysis*, Basel–Berlin–Boston, Birkhäuser (1990).
- [4] AVERNA D., *Lusin Type Theorems for Multifunctions, Scorza Dragoni's Property and Carathéodory Selections*, Boll. Unione Mat. Ital. (7) 8-A (1994), 193-202.
- [5] BADER R., KRYSZEWSKI W., *On the Solution Sets of Differential Inclusions And the Periodic Problem in Banach Spaces*, Nonlinear Anal. 54 (2003), 707-754.
- [6] BOTHE D. *Flow Invariance for Perturbed Nonlinear Evolution Equations*, Abstr. Appl. Anal. 1 (1996), 417-433.

- [7] BOTHE D., *Nonlinear Evolutions in Banach Spaces – Existence and Qualitive Theory with Applications to Reaction-Diffusion Systems*, Habilitationsschrift (1999), http://www.csi.tu-darmstadt.de/media/csi/staff/downloads_1/habilschrift.pdf.
- [8] BREZIS H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York, Springer (2011).
- [9] CARJA O., NECULA M., VRABIE I. I., *Viability, Invariance and Applications*, Elsevier Science B. V., Amsterdam 2007.
- [10] CHEN D.-H., WANG R.-N., ZHOU Y., *Nonlinear evolution inclusions: Topological characterizatiob of solutions and applications*, J. Funct. Anal. 265 (2013), 2039-2073.
- [11] ĆWISZEWSKI A., *Topological Degree Methods for Perturbations of Operators Generating Compact C_0 Semigroups*, J. Differential Equations 220 (2006), 434-477.
- [12] DIESTEL J., RUESS W. M., SCHACHERMAYER W. *Weak Compactness in $L^1(\mu, X)$* , Proc. Amer. Math. Soc. 118 no. 2 (1993), 447-453.
- [13] GABOR D., KRYSZEWSKI W., *A Global Bifurcation Index for the Set-Valued Perturbations of Fredholm Operators*, Nonlinear Anal. 73 (210), 2714-2736.
- [14] GABOR D., KRYSZEWSKI W., *The Generalized Krasnosel'skii Formula and Bifurcation Problem*, preprint.
- [15] GÓRNIOWICZ L., *Topological Fixed Point Theory of Multivalued Mappings*, Dordrecht, Springer (2006).
- [16] GÓRNIOWICZ L., KRYSZEWSKI W., PLASKACZ S., *Juliusz Schauder Center Winter School on Methods in Multivalued Analysis*, Toruń, Juliusz Schauder Center for Nonlinear Studies, Nicolaus Copernicus University (2006).
- [17] GRANAS A., DUGUNDJI J., *Fixed Point Theory*, Berlin, Springer (2003).
- [18] HANDBOOK OF *Multivalued Analysis, Volume I: Theory*, Dordrecht, Kluwer, The Netherlands (1997).
- [19] HYMAN D. M., *On Decreasing Sequence of Compact Absolute Retracts*, Fund. Math. 64 (1969), 91-97.
- [20] JUNIEWICZ M., NGUYEN H. T., ZIEMIŃSKA J., *Carathéodory CM-selectors for Oppositely Semicontinuous Multifunctions of Two Variables*, Bull. Pol. Acad. Sci. Math. 50 no. 1 (2002), 47-57.
- [21] KAMENSKII M., OBUKHOVSKII V., ZECCA P., *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter, Berlin (2001).
- [22] KANIGOWSKI A., KRYSZEWSKI W., *Perron-Frobenius and Krein-Rutman theorem for tangentially positive operators*, Cntr. Eur. J. Math., 10 (2012) 2240-2263.
- [23] KRASNOSEL'SKII M.A., ZABREIKO P.P., *Geometrical Methods of Nonlinear Analysis*, Berlin–Heidelberg–New York–Tokyo, Springer–Verlag (1984).
- [24] KUCIA A., *Scorza Dragoni Type Theorems* Fund. Math. 138 (1991) 197-203.
- [25] LACHER R. C., *Cell-loke mappings and their generalizations*, Bull Amer. Math. Soc. 83 (1977), 336-552.
- [26] PAVEL N. H., *Invariant sets for a class of semi-linear equations of evolution*, Nonl. Anal. 1 (1977) 187-196.
- [27] PAVEL N. H., *Differential Equations, Flow Invariance and Applications*, Res. Notes Math. 113, Pitman 1984.
- [28] PAZY A., *Semigroups of Linear Operator and Applications to Partial Differential Equations*, New York, Springer–Verlag (1983).
- [29] SHOWALTER R. E., *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surv. and Mon. 49, American Math. Soc., Providence 1997.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCES, NICOLAUS COPERNICUS UNIVERSITY

E-mail address: wkrysz@mat.umk.pl, jsiem@mat.umk.pl